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Information Sciences xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

A note on modal logic and possibility theory ☆

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ARTICLE INFO

Article history:
Received 22 January 2014
Received in revised form 2 April 2014
Accepted 8 April 2014
Available online xxxx

Keywords:
Modal logic
Possibility theory
FS-model

ABSTRACT

There are two theories in which the concept of possibility plays an important role—modal logic and possibility theory. The roles are different, and so are the agendas of modal logic and possibility theory. To gain an insight into the differences, a very simple model of modal logic is constructed. The model has the structure of a finite-state system, referred to as the FS-model. The FS-model may be viewed as a simple interpretation of Kripke model—an interpretation which is easy to understand. The FS-model is in the spirit of graph models of modal logic. The FS-model readily lends itself to generalization. Concrete versions of the FS-model serve as examples.

1. Introduction

There are two theories in which the concept of possibility plays an important role—modal logic and possibility theory. The role of the concept of possibility in modal logic is very different from its role in possibility theory. Interestingly, on a deeper level, a striking similarity comes to light. In large measure, what follows is motivated by the question: In what basic ways does the concept of possibility in modal logic differ from the concept of possibility in possibility theory?

Modal logic is a deep theory which is not easy to understand. [2] For comparison of modal logic with possibility theory, what is constructed in this note is a very simple abstract model which has the structure of a finite-state system, referred to as the FS-model. The best known model of modal logic is Kripke model. There are many models which are equivalent to Kripke model. [1,7] The FS-model may be viewed as a simple interpretation of Kripke model and is in the spirit of graph models of modal logic. The FS-model is easy to understand and readily lends itself to generalization. A summary of the FS-model is described in the following. It should be underscored that this note touches upon only elementary aspects of modal logic and possibility theory.

2. FS-model

The FS-model has five principal components.

(1). A collection of states, $W = (w_1, \ldots, w_n)$. W is referred to as the state space of FS. In the abstract model, the states are simply symbols with no meaning. The states do have meaning in concrete versions of the FS-model.

☆ Research supported in part by ONR N00014-02-1-0294, Omron Grant, Tekes Grant, Azerbaijan Ministry of Communications and Information Technology Grant, Azerbaijan University of Azerbaijan Republic and the BISC Program of UC Berkeley.
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http://dx.doi.org/10.1016/j.ins.2014.04.002

(2). A collection of collections of inputs, with each state, w_i, associated with a collection of inputs, $U_i = (u_{i1}, \ldots, u_{ik})$, with k dependent on w_i, $k = k(i)$. (Fig. 1)
(3). State-transition function, f,

$$s_{t+1} = f(s_t, u_t),$$

where s_t is the state at time t, s_{t+1} is the next state, and u_t is the input at time t. s_t and s_{t+1} take values in the state space, W. The state-transition function is represented as a state diagram (Fig. 2).
(4). A proposition, p. A truth function, tr(p, w_i), associates with each state, w_i, the truth value, t_i, of p in w_i, t_i = Tr(p, w_i). If p is a crisp proposition, t_i is either true (1) or false (0). If p is a fuzzy proposition, t_i takes values in the unit interval.
(5). A target set, $T(p)$, is a collection of what are called target states. A target state, w_j, is a state in which $t_j = 1$. Thus, $T(p)$ is the collection of all states in which p is true. The target set is defined by p, that is, p serves to define the target set, $T(p)$. A state, w_j, satisfies $T(p)$, if w_j is a target state. Thus, $T(p) = \{w_j | \text{tr}(p, w_j) = 1\}$. The target set for not p is the complement in W of the target set for p (Fig. 3).

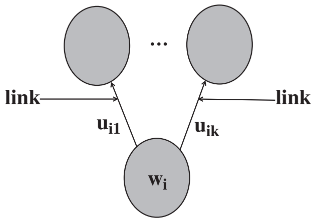

Fig. 1. Inputs in state w_i.

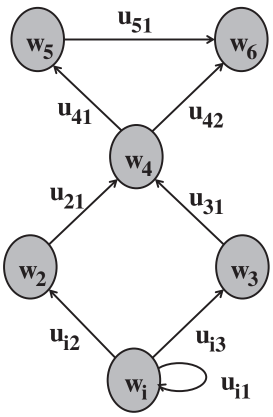

Fig. 2. State-transition diagram.

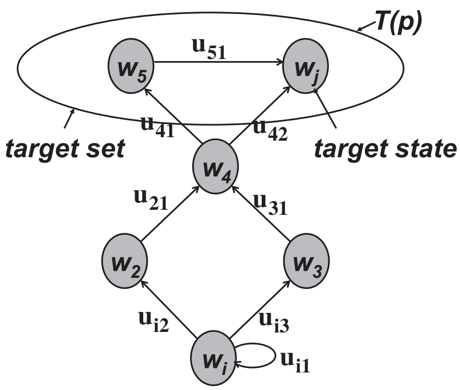

Fig. 3. Target set and target states. $T(p)$ is defined by p.

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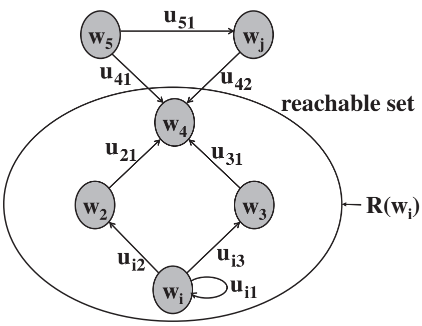

Fig. 4. Reachable set, $R(W_i)$.

Definition 1. Reachability. A state, w_j, is reachable from state, w_i, if there is a sequence of inputs which takes w_i into w_j. Example. In Fig. 4, $u_{i3}\ u_{31}$ takes w_i into w_4.

Definition 2. Reachable set. Each state, w_i, is associated with the set of all states, $R(w_i)$, which are reachable from w_i. $R(w_i)$ is referred to as the reachable set from w_i. That is, $R(w_i) = \{w_j | w_j$ is reachable from $w_i\}$ (Fig. 4).

Definition 3. Possibility. p is possible in state w_i, abbreviated as possible p/w_i, if the intersection of $R(w_i)$ and $T(p)$ is not empty, that is, if there exists a target state in $T(p)$ which is reachable from w_i.

Definition 4. Impossibility. p is not possible (impossible) in state w_i if the intersection of $R(w_i)$ and $T(p)$ is empty, that is, if there is no target state which is reachable from w_i.

Definition 5. Necessity (certainty). p is necessary (certain) in state w_i, abbreviated as necessary p/w_i, if $R(w_i)$ is contained in $T(p)$, that is, if all reachable states are target states. In Fig. 5, possibility, impossibility and necessity are represented as three Venn diagrams.

Venn diagrams provide a simple way of representing definitions, axioms and simple theorems. As an example, Fig. 6 shows how Venn-diagram-based reasoning leads to the basic relation

$$\text{possible } p/w_i = \text{not necessary(not } p)/w_i.$$

Observation. What is described above is a summary of the FS-model. Clearly, the FS-model is much simpler and much easier to understand than Kripke model. Compare the two definitions of possibility.

Kripke model definition. p possible is true in possible world w_i if p is true in a possible world which is accessible from w_i.

FS-model definition. p is possible in state w_i if the target set, $T(p)$, is reachable from w_i, that is, if a target state is reachable from w_i.

The concepts of possible world and accessibility in Kripke model are much less transparent than the concepts of state and reachability in the FS-model.

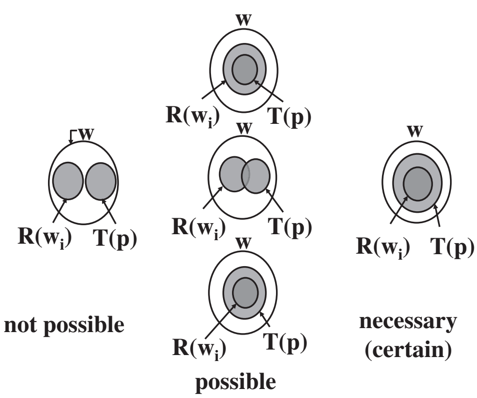

Fig. 5. Three basic configurations. Venn diagrams for not possible, possible and necessary (certain).

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The FS-model can readily be generalized. A natural generalization is one in which p is a fuzzy proposition, e.g., economy is good. A fuzzy proposition, p, defines the target set, $T(p)$, as a fuzzy set in which the grade of membership of w_j in $T(p)$ is the truth value of p in state w_j, that is, $\mu_{T(p)}(w_j) = \text{tr}(p, w_j) = t_i$.

Example of a concrete version of the FS-model. The road map of the United States. Assume that the state space, W, consists of cities in the United States; the state-diagram is the road map, and the inputs are names or numbers of roads. Assume that p is the proposition, p: I am in Pennsylvania. The target set, $T(p)$, consists of cities in Pennsylvania. Assume that w_i is San Francisco. Let R(SF) be the set of cities which are reachable from San Francisco. The proposition, p: I am in Pennsylvania, is possible in San Francisco if a city in Pennsylvania is reachable from San Francisco.

As a generalization, assume that p is the proposition, p: I am in Pennsylvania or vicinity. In this case, $T(p)$ is the fuzzy set of cities which are in Pennsylvania or vicinity. Assume that the grade of membership of a city, w_j, in $T(p)$ is μ_j. Then, the possibility of p: I am in Pennsylvania or vicinity, starting in San Francisco, is the largest μ_j in the intersection of R(SF) and $T(p)$. It should be noted that the generalized FS-model described above may be viewed as a model of fuzzy modal logic [3,5,6].

Example. Medical treatment. Assume that I am a patient in a hospital. I am sick. My state is described by a collection of parameters. An input is a medical procedure or treatment. p: I am cured. The target set, $T(p)$, consists of states in which p: I am cured, is true. In the initial state, I am sick, p is possible if there is a sequence of treatments which take the initial state, w_i: I am sick, into a target state, w_j: I am cured.

Generalization. Fuzzy state-transition function. A simple version is one in which each link in the state-diagram is associated with a weight which is a number in the unit interval. In this case, reachability becomes a matter of degree. More concretely, let v_{ij} be the conjunction of weights along a path from w_i to w_j. The degree of reachability of w_j from w_i is defined as the supremum of the v_{ij} over all paths from w_i to w_j. The degree of reachability defines the reachable set, $R(w_i)$, as a fuzzy set. The grade of membership of w_j in $T(p)$ is the degree of satisfiability of w_j in $T(p)$. Let S be the intersection of $R(w_i)$ and $T(p)$. S is a fuzzy set in which the grade of membership, $\mu_S(w_j)$, is the conjunction of the grade of membership, $\mu_{R(w_j)}$ and the grade of membership of w_j in $T(p)$, $\mu_{T(p)}(w_j)$ (the degree of satisfiability of w_j in $T(p)$) (Fig. 7).

Thus, in the FS-model in which the reachable set and the target set are fuzzy sets, we have,

Definition. The possibility of p in state w_i is the supremum of $\mu_S(w_j)$ over all states in the intersection of $R(w_i)$ and $T(p)$. Equivalently, degree of possibility of p in $w_i = \sup_S$ (degree of reachability of w_j from w_i and degree of satisfiability of w_j in $T(p)$), where and = conjunction. The models sketched above are abstract models of fuzzy modal logic.

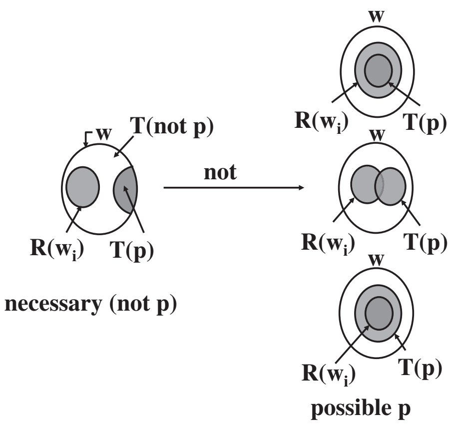

Fig. 6. Venn-diagram-based reasoning.

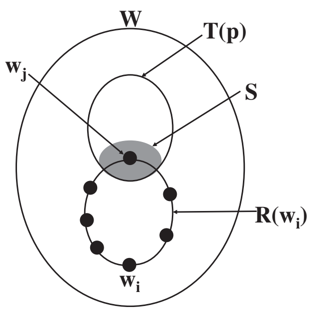

Fig. 7. Reachability is a matter of degree.

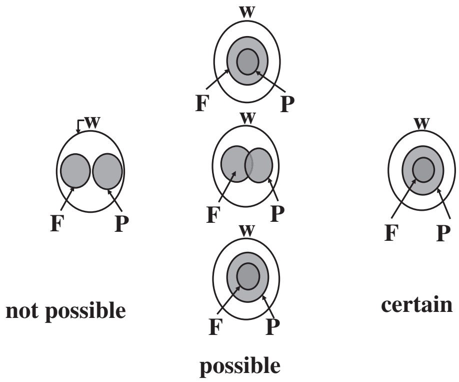

Fig. 8. Three basic configurations. Venn models for not possible, possible and certain.

What should be underscored is that what is described above is a brief summary of the FS-model and its basic generalizations.

3. Basics of possibility theory

In my 1978 paper on possibility theory [8], possibility theory is based on the theory of fuzzy sets. Extensive contributions to the development of possibility theory have been made by Dubois and Prade. [4] Following is a brief synopsis of the theory.

Let p be a proposition, possibly drawn from a natural language. The point of departure in possibility theory is the question: What is the possibility of p, with the understanding that possibility is a matter of degree, with degrees taking values in the unit interval? Informally, the possibility of p is the degree of consistency of p with factual information, F. Assume that p is represented as a fuzzy proposition of the form

$$X \text{ is } P,$$

where X takes values in a space, U, and P is a fuzzy set in U, with a specified membership function μ_P. Similarly, assume that F is represented as a fuzzy proposition of the form

$$X \text{ is } F,$$

where F is a fuzzy set in U, with a specified membership function, μ_F. Let u be a generic value of X. Denote the possibility that $X = u$ as Poss($X = u$).

Definition. Poss($X = u$) is defined as the grade of membership of u in P

$$\text{Poss}(X = u) = \mu_P(u).$$

Definition. The possibility distribution, Poss(P), is defined as

$$\text{Poss}(P) = \mu_P.$$

What this means is that P plays the role of the possibility distribution of X.

Definition. The possibility measure of P given F, Poss($P|F$) is defined as

$$\text{Poss}(P|F) = \text{sup}_u(F \cap P)$$

or, more concretely,

$$\text{Poss}(P|F) = \text{sup}_u(\mu_F(u) \text{ and } \mu_P(u)),$$

where and = conjunction. Equivalently, Poss($P|F$) is the conditional possibility measure of P given F.

Example (*crisp*). Let X = Birthplace of Ron Yager. F = Cities in the United States. Then,

$$\text{Poss}(\text{Birthplace of Ron Yager} = \text{Boston}) \text{ is } 1.$$

In words, given that Ron Yager was born in the United States, and that Boston is in the United States, "It is possible that Ron Yager was born in Boston."

Example (*fuzzy*). F = Cities in the vicinity of New York. The grade or membership of Larchmont in F is 0.9. In this case,

$$\text{Poss(Birthplace of Ron Yager} = \text{Larchmont) is 0.9.}$$

In words, given that Ron Yager was born in the vicinity of New York, it is 0.9 possible that Ron Yager was born in Larchmont.

For the purpose of comparison with the FS-model it is helpful to restate the definitions given above in a different form.

Definition (*Crisp F and P*). p is possible if the intersection of F and P is not empty.

Definition. p is not possible if the intersection of F and P is empty.

Definition. p is certain if F is contained in P. The Venn diagrams (configurations) of these definitions are shown in Fig. 8.

Possibility theory provides a mathematically well-defined conceptual framework for dealing with possibilities. In this sense, possibility theory complements modal logic.

There are significant differences as well as similarities between the roles which the concept of possibility plays in modal logic and possibility theory. In modal logic, possibility relates to the possibility of reaching a target set. In possibility theory, if p is interpreted as a goal, P, possibility is focused on the relationship between factual information and the goal, P. In this perspective, what is a striking similarity between modal logic and possibility theory is that the Venn diagrams associated with not possible, possible and necessary (certain) are identical modulo labeling of sets. One significant difference is that in modal logic we have a finite-state system in which there are transitions from one state to another. In possibility theory, there is no dynamics. Basically, modal logic and possibility theory are complementary rather than competitive. Modal logic is a more specialized theory and has fewer applications than possibility theory. Possibility theory, in conjunction with probability theory, plays an important role in dealing with uncertainty. Possibility theory adds to probability theory an important capability—the capability to compute with information which is described in natural language.

4. Concluding remark

The FS-model is very simple and very easy to understand. The conceptual structure of the FS-model readily lends itself to generalization. The FS-model serves as a bridge between modal logic and possibility theory.

Acknowledgments

To Kazem Sadegh-Zadeh, Francesc Esteva and Luis Godo.

References

[1] Gerard Allwein, J. Michael Dunn, Kripke models for linear logic, J. Symb. Log. 58 (2) (1993) 514–545.
[2] M.J. Cresswell, G.E. Hughes, A New Introduction to Modal Logic, Routledge, New York, 1996.
[3] A.A. Dodó, J. Marcos, Fuzzy modal logics of confluence, in: L.C. Barros, B.R.C. Bedregal, J. Marcos, J.A.F. Roveda, R.H.N. Santiago, W. Seixas, (Eds.), Recentes Avanços em Sistemas Fuzzy, SBMAC, 2012, pp. 328–337.
[4] D. Dubois, H. Prade, Possibility Theory, Plenum Press, New-York, 1988.
[5] P. Hájek, D. Harmancova, F. Esteva, P. García, L. Godo, On modal logics for qualitative possibility in a fuzzy setting, in: R. López de Mántaras, D. Poole (Eds.), Uncertainty in Artificial Intelligence, Morgan Kaufmann Publishers, San Francisco, USA, 1994, pp. 278–285.
[6] A. Mironov, Fuzzy modal logics, J. Math. Sci. 128 (6) (2005) 3461–3483.
[7] J. Sowa, Worlds, models, and descriptions, studia logica, Spec. Issue Ways Worlds II 84 (2) (2006) 323–360.
[8] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.